Quantum mechanics II, Chapter 4 : Reduced States and Decoherence

TA : Achille Mauri, Behrang Tafreshi, Gabriel Pescia, Manuel Rudolph, Reyhaneh Aghaei Saem, Ricard Puig, Sacha Lerch, Samy Conus, Tyson Jones

Our systems do not always stay closed.

```
  ,-.       _,---._ __  / \
 /  )    .-'       `./ /   \
(  (   ,'            `/    /|
 \  `-"             \'\   / |
  `.              ,  \ \ /  |
   /`.          ,'-`----Y   |
  (   x        ;        |   '
  |  ,-.    ,-'         |  /
  |  | (   |            | /
  )  |  \  `.___________|/
  `__'   `__'
```

Problem 1 : No signalling

Use your new-found understanding of reduced states to justify the no signalling principle (i.e. to argue why it is not possible to use entanglement to signal faster than light). Discuss with a partner.

Problem 2 : Density operator, partial trace, information and measurement

Alice and Bob share state

$$|\psi\rangle = \frac{|01\rangle - |10\rangle}{\sqrt{2}} \tag{1}$$

1. What is the density matrix of the system $\hat{\rho}$ with 2 qubits ?
2. Verify that it is a pure state by calculating $\mathrm{Tr}(\hat{\rho}^2)$.

Note $\hat{\rho}_B = \mathrm{Tr}_A\,\hat{\rho}$ the density matrix obtained by partial trace on Alice's qubit. This matrix is an operator on the Hilbert space of the second qubit (Bob's), and reflects the information available to Bob.

3. Calculate $\hat{\rho}_B$ and link that result to the probability Bob has to get outcome 0 or 1 when he measures his qubit in the computational basis (We will write $\hat{O}$ the corresponding observable). Also verify that we have $\langle\hat{O}\rangle = \mathrm{Tr}[\hat{\rho}(\mathbb{1}\otimes\hat{O})] = \mathrm{Tr}(\hat{\rho}_B\hat{O})$.
4. Does the matrix ρ_B describe a pure state of the second qubit ? Justify by calculating $\mathrm{Tr}(\hat{\rho}_B^2)$. What about if the 2 qubit state $|\psi\rangle$ being separable in the form $|\psi_A\rangle \otimes |\psi_B\rangle$? We sometimes say that statistical mixtures of the state of a system is the fruit of entanglement of this system with its environment ; how can we interpret this in the light of the previous results ?

We admit that when the measurement of an observable $\hat{M}$ on the system gives the result m, then the density matrix ($\hat{\rho}$ before measurement) reads

$$\hat{\rho}' = \frac{\hat{P}_m\hat{\rho}\hat{P}_m^\dagger}{\mathrm{Tr}(\hat{P}_m^\dagger\hat{P}_m\hat{\rho})}, \tag{2}$$

where $\hat{P}_m$ is the projector on the subspace relative to m.

Disclaimer : If you don't remember why this is the case have a look at the measurement postulate and the post-measurement state `https://en.wikipedia.org/wiki/Measurement_in_quantum_mechanics#State_change_due_to_measurement`

5. What is the state with 2 qubits $|\psi'\rangle$ obtained when Alice measure her qubit in the computational basis and finds 0 ? Compare $|\psi'\rangle\langle\psi'|$ and $\hat{\rho}'$.
6. When Alice measures her qubit on state $|\psi\rangle$ and finds 0, what is the density matrix ? $\hat{\rho}'_B$? Comment.

Problem 3 : Environment induced decoherence

1. Consider a composite system that is prepared in the initial state $|\psi\rangle = \sum_j c_j |E_j\rangle_A \otimes |\phi\rangle_B$ and evolves under a Hamiltonian $H_{AB} = \sum_j |E_j\rangle\langle E_j|_A \otimes H_B^{(j)}$ for time t.
   Find an expression for the reduced states $\rho_A(t)$ and $\rho_B(t)$ of systems A and B as a function of time.
2. Under what circumstances do A and B remain pure for all times ?
3. Under what circumstances does $\rho_A(t)$ become approximately diagonal in the basis $\{|E_j\rangle\}$?
4. Why do you think $H_{AB} = \sum_j |E_j\rangle\langle E_j|_A \otimes H_B^{(j)}$ is sometimes described as quantifying the "*measurement limit* of system-environment interactions" ? (Bonus question (discuss with a partner) : how is this related to the measurement problem ?)

Problem 4 : Decoherence and dephasing of a single qubit

1. Consider applying a random R_z rotation, i.e. $e^{-i\vartheta/2\sigma_z}$ for $\vartheta/ \in [-\pi, \pi]$ to a generic initial pure qubit state $|\psi\rangle = \sin(\theta)|0\rangle + \cos(\theta)e^{-i\phi}|1\rangle$. What is the resulting mixed state on average ? Sketch this on the Bloch sphere.

   Disclaimer : What does "average mixed state" even mean ? In this case, it means that we take the average over all states with respect to this σ_z rotation. How do we take the average ? We just compute
   $$\rho_{\text{av}} = \frac{1}{2\pi} \int_{-\pi}^{\pi} R_z(\vartheta) |\psi\rangle \langle\psi| R_z^\dagger(\vartheta) d\vartheta \tag{3}$$
2. Consider now applying a random R_z rotation and then a random R_x rotation. What is the resulting mixed state on average ? And what if you now apply all three (a random R_z, R_x and R_y) ? Sketch this on the Bloch sphere.
3. What if instead you apply a random R_z rotation with probability p and do nothing with probability $1-p$? And what if you apply a random R_z rotation then a random R_x rotation with probability p, and do nothing with probability $1-p$? Sketch this on the Bloch sphere.
4. Suppose you now instead throw away your initial state and prepare the maximally mixed state $\mathbb{I}/2$ with probability p and do nothing with probability $1-p$? Sketch this on the Bloch sphere.
5. What do you conclude from all this ?